Topology and its Applications 157 (2010) 1607–1621

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Properties of comultiplications on a wedge of spheres

Martin Arkowitz[a], Dae-Woong Lee[b,*]

[a] Department of Mathematics, Dartmouth College, Hanover, NH 03755, United States

[b] Department of Mathematics, and Institute of Pure and Applied Mathematics, Chonbuk National University, Jeonju, Jeonbuk 561-756, Republic of Korea

ARTICLE INFO

Article history:
Received 25 November 2009
Received in revised form 9 March 2010
Accepted 9 March 2010

MSC:
primary 55Q25
secondary 55Q15, 55P40

Keywords:
Co-H-spaces
Comultiplications
Homotopy associativity
Homotopy commutativity
Whitehead products
Hopf–Hilton invariants
Wedges of spheres
Algebraic loops
Inversivity
Power-associativity
The Moufang property

ABSTRACT

In this paper we study the set of comultiplications on a wedge of a finite number of spheres. We are interested in group theoretic properties of these comultiplications such as associativity and commutativity and loop theoretic properties such as inversivity, power-associativity and the Moufang property. Our methods involve Whitehead products in wedges of spheres and the Hopf–Hilton invariants. We obtain extensive results for a restricted class of comultiplications, namely, the one-stage quadratic or cubic comultiplications.

1. Introduction

Spaces with a comultiplication, also called co-H-spaces, are of fundamental importance in the study of homotopy theory. The main reason for this is that any two homotopy classes of maps from a co-H-space X to any space Y can be added together to yield a homotopy class from X to Y. This gives a natural binary operation with identity on this set of homotopy classes which has the structure of an algebraic loop (see Section 6). If the comultiplication is associative, then this algebraic loop is a group, and the group operation depends on the comultiplication of X. An important class of co-H-spaces consists of spaces which are suspensions, in particular, all n-spheres, $n \geqslant 1$. It is the associative comultiplication of the n-sphere S^n which induces group structure on the nth homotopy group of Y, the set of homotopy classes of S^n into Y. For $n \geqslant 2$, this comultiplication is unique and commutative up to homotopy (which is the reason that the nth homotopy group is abelian). It is easily seen that the wedge of a finite number of co-H-spaces is a suspension and thus has a suspension comultiplication. It is therefore natural to ask about the comultiplications on a wedge of t spheres, $t \geqslant 2$. This set of comultiplications is complex – it often contains many comultiplications which have many different properties. In this paper we examine these comultiplications and some of their properties.

* Corresponding author. Tel.: +82 63 270 3367; fax: +82 63 270 3363.
E-mail addresses: Martin.A.Arkowitz@Dartmouth.edu (M. Arkowitz), dwlee@jbnu.ac.kr (D.-W. Lee).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2010.03.001

The study of comultiplications of a co-H-space has been carried out by several authors. In [2] comultiplications on a wedge of circles were investigated by using the methods of free groups. In [7] the comultiplications on a wedge of two Moore spaces were considered by means of homological algebra. In [4] and [5] rational homotopy techniques were applied to obtain results for rational co-H-spaces. Furthermore, in [8] and [9] the space used to define the Fox homotopy groups was studied and shown to have the homotopy type of a wedge spheres. However, the natural comultiplication for this space is not the suspension comultiplication.

In our paper [3] we examined comultiplications on a wedge of two spheres. Here we use similar methods to explore the more difficult problem of comultiplications on a wedge of an arbitrary number of spheres. We consider group theoretic properties such as associativity and commutativity as well as loop theoretic properties such as inversivity, power-associativity and the Moufang property. The Eckmann–Hilton dual of a co-H-space is an H-space, and group theoretic and loop theoretic properties of multiplications on an H-space have been widely studied (see [13] and [6]).

The paper is organized as follows: In Section 2 we give some elementary results on the group structure on the set of comultiplications of any cogroup (that is, associative co-H-space with inverses). In Section 3 and the rest of paper we consider the cogroup X which is a wedge of t spheres and introduce our basic tools, namely, Hilton's Theorem and the Hopf–Hilton invariants. Section 4 is devoted to a study of associativity. After some generalities, we consider one-stage comultiplications on X, that is, comultiplications which are trivial when restricted to all but one sphere in the wedge. For these we obtain extensive results on associativity when the non-trivial part of the comultiplication consists of 2-fold Whitehead products (quadratic comultiplications) or of 3-fold Whitehead products (strictly cubic comultiplications). For these comultiplications we study commutativity in Section 5. Finally, in Section 6 we consider loop properties of the comultiplications mentioned above. We obtain general results for inversivity and power-associativity for any one stage comultiplication and more specific results on all of our loop properties for a restricted class of comultiplications.

2. Comultiplications

In this paper all spaces are based and have the based homotopy type of based, connected CW-complexes of finite type (i.e., with finitely many cells in each dimension). All maps and homotopies preserve the base point. We do not distinguish notationally between a map and its homotopy class. Thus we write $f = g$ to mean that the homotopy classes f and g are equal or that the maps f and g are homotopic (also written $f \simeq g$).

A pair (Y, φ) consisting of a space Y and a map $\varphi : Y \to Y \vee Y$ is called a **co-H-space** if $p_1\varphi = 1$ and $p_2\varphi = 1$, where p_1 and p_2 are the projections $Y \vee Y \to Y$ onto the first and second summands of the wedge and 1 is the identity map of Y. Then $\varphi : Y \to Y \vee Y$ is called a **comultiplication**. Equivalently, (Y, φ) is a co-H-space if $j\varphi = \Delta : Y \to Y \times Y$, where Δ is the diagonal map and $j : Y \vee Y \to Y \times Y$ is the inclusion. A comultiplication φ is called **associative** if $(\varphi \vee 1)\varphi = (1 \vee \varphi)\varphi : Y \to Y \vee Y \vee Y$, and φ is called **commutative** if $T\varphi = \varphi$, where $T : Y \vee Y \to Y \vee Y$ is the switching map. A **left inverse** for φ is a map $L : Y \to Y$ such that $\nabla(L \vee 1)\varphi = 0$ and a **right inverse** is a map $R : Y \to Y$ such that $\nabla(1 \vee R)\varphi = 0$, where $\nabla : Y \vee Y \to Y$ is the folding map and 0 is the constant map. If (Y, φ) is an associative co-H-space with a left and right inverse, then (Y, φ) is called a **cogroup** and the comultiplication φ is a **cogroup comultiplication**.

If (Y, φ) is a co-H-space and Z is any space, then the set $[Y, Z]$ of homotopy classes of maps of Y into Z is a semi-group with unit 0, the constant map. The binary operation, denoted '+', is defined for $a, b \in [Y, Z]$ by the composition

$$Y \xrightarrow{\varphi} Y \vee Y \xrightarrow{a \vee b} Z \vee Z \xrightarrow{\nabla} Z.$$

If $f : Z \to Z'$ is a map, then the induced map $f_* : [Y, Z] \to [Y, Z']$ is a homomorphism. It is easily seen that (1) if φ is commutative, then $[Y, Z]$ is commutative, (2) if φ is associative, then $[Y, Z]$ is associative and (3) if L is a left inverse for φ, then every element $a \in [Y, Z]$ has a left inverse (defined as the composition aL), and similarly for right inverses. Furthermore, if φ is associative or commutative, then every left inverse is a right inverse (and every right inverse is a left inverse). In particular, a cogroup comultiplication φ on Y induces group structure on $[Y, Z]$, for every space Z, such that f_* is a group homomorphism.

Note that in the semi-group $[Y, Y \vee Y]$, the comultiplication $\varphi = i_1 + i_2$, where $i_1, i_2 : Y \to Y \vee Y$ are the two inclusions. Moreover, in the semi-group $[Y, Y]$, we have $L + 1 = 0$ and $1 + R = 0$. For more detailed information about co-H-spaces, see the survey article [1].

For the remainder of this section, let (Y, φ_0) be a cogroup. We will consider the collection of all comultiplications on Y and obtain some general results. In subsequent sections we will specialize Y to be a wedge of spheres to obtain more concrete results.

Let $\mathcal{C}(Y)$ be the set of all (homotopy classes of) comultiplications on Y and let $\mathcal{C}_c(Y) \subseteq \mathcal{C}(Y)$ be the commutative comultiplications.

Proposition 2.1. *If (Y, φ_0) is a cogroup and a finite 1-connected CW complex, then $\mathcal{C}(Y)$ is a finitely-generated group. If, in addition, (Y, φ_0) is commutative, then $\mathcal{C}(Y)$ is a finitely-generated abelian group with subgroup $\mathcal{C}_c(Y)$.*

Proof. We first define the group structure in $\mathcal{C}(Y)$ as follows: Let $Y\flat Y$ be the homotopy fiber of $j: Y\vee Y\to Y\times Y$. Then $Y\flat Y$ is the space $E(Y\times Y; Y\vee Y, *)$ of paths in $Y\times Y$ which begin in $Y\vee Y$ and end at the base point $*$. Let $i: E(Y\times Y; Y\vee Y, *)\to Y\vee Y$ be the map defined by $i(l)=l(0)$. Note that

$$Y\flat Y \xrightarrow{i} Y\vee Y \xrightarrow{j} Y\times Y$$

is essentially a fiber sequence and hence it follows that there is an exact sequence of groups

$$\cdots \xrightarrow{(\Omega j)_*} [Y, \Omega(Y\times Y)] \longrightarrow [Y, Y\flat Y] \xrightarrow{i_*} [Y, Y\vee Y] \xrightarrow{j_*} [Y, Y\times Y].$$

Then j_* is onto for all Y and so $(\Omega j)_*$ is onto since $(\Omega j)_*: [Y,\Omega(Y\vee Y)]\to [Y,\Omega(Y\times Y)]$ corresponds to j_*: $[\Sigma Y, Y\vee Y]\to[\Sigma Y, Y\times Y]$. Therefore we obtain an exact sequence of groups

$$0 \longrightarrow [Y, Y\flat Y] \xrightarrow{i_*} [Y, Y\vee Y] \xrightarrow{j_*} [Y, Y\times Y] \longrightarrow 0,$$

where the group operations are induced from the cogroup structure (Y,φ_0). If $\varphi\in\mathcal{C}(Y)$, then $j_*(-\varphi_0+\varphi)=0$, and so $\varphi=\varphi_0+i\alpha$, for some unique $\alpha\in[Y, Y\flat Y]$. In this way we obtain a bijection $\theta:\mathcal{C}(Y)\to[Y, Y\flat Y]$ defined by $\theta(\varphi)=\alpha$. Since $[Y, Y\flat Y]$ is a group, we give group structure to $\mathcal{C}(Y)$ by requiring θ to be an isomorphism. Clearly φ_0 is the identity of the group $\mathcal{C}(Y)$. It follows from the hypothesis that $[Y, Y\flat Y]$ is finitely-generated, and so $\mathcal{C}(Y)$ is a finitely-generated group. Clearly $\mathcal{C}(Y)$ is abelian if (Y,φ_0) is a commutative cogroup.

Next we assume that φ_0 is commutative and consider commutative comultiplications. If $T: Y\vee Y\to Y\vee Y$ is the switching map, then T induces a map $\tau: Y\flat Y\to Y\flat Y$ such that $i\tau=Ti: Y\flat Y\to Y\vee Y$. We have

$$\begin{aligned} T\varphi &= T(\varphi_0+i\alpha)\\ &=\varphi_0+Ti\alpha\\ &=\varphi_0+i\tau\alpha. \end{aligned}$$

Therefore φ is commutative $\iff T\varphi=\varphi \iff i\tau\alpha=i\alpha \iff \tau\alpha=\alpha$, since $i_*:[Y, Y\flat Y]\to[Y, Y\vee Y]$ is a monomorphism. Thus $\mathcal{C}_c(Y)$ corresponds under θ to $\{\alpha\in[Y, Y\flat Y] \mid \tau(\alpha)=\alpha\}$, and so $\mathcal{C}_c(Y)$ is a subgroup of $\mathcal{C}(Y)$. □

Let us denote the cardinality of any set S by $|S|$. Let $N=|\mathcal{C}(Y)|=|[Y, Y\flat Y]|$ and $N_c=|\mathcal{C}_c(Y)|$. Then we have shown the following.

Proposition 2.2. *If (Y,φ_0) is a commutative cogroup, then $N_c \mid N$.*

In this notation, we allow any positive number (or ∞) to divide ∞.

3. Wedges of spheres

In this section and the rest of the paper we restrict attention to comultiplications on wedges of spheres. This enables us to obtain precise and detailed results on properties of comultiplications. We will use the following notation in the rest of the paper.

- X will always denote the wedge of spheres $S^{n_1}\vee S^{n_2}\vee\cdots\vee S^{n_t}$, where $t\geqslant 2$ and $1<n_1<\cdots<n_t$.
- $k_j: S^{n_j}\to X$ will be the jth inclusion for $j=1,2,\ldots,t$.

Since any comultiplication φ on X is completely determined by the elements $\varphi k_j\in\pi_{n_j}(X\vee X)$, it is necessary to know these latter groups. For this we next recall results of Hilton [10].

We consider a wedge of spheres $W=S_1\vee\cdots\vee S_k$, where $S_j=S^{r_j}$, and let $i_j: S_j\to W$ be the inclusion, $j=1,\ldots,k$. We will form Whitehead products of these elements.

We inductively define and order **basic (Whitehead) products** as follows: Basic products of weight 1 are (in order) $i_1,\ldots,i_k$. Assume basic products of weight $<n$ have been defined and ordered so that if $r<s<n$, any basic product of weight r is less than all basic products of weight s. Then a basic product of weight n is a Whitehead product $[a,b]$, where a is a basic product of weight m and b is a basic product of weight l, $m+l=n$, $a<b$. Furthermore, if b is a Whitehead product $[c,d]$ of basic products c and d, then we require that $c\leqslant a$. The basic products of weight n are ordered arbitrarily among themselves and are greater than any basic product of weight $<n$. Note that to a basic product of weight n we can associate a string of distinct symbols $i_{v_1},\ldots,i_{v_s}$, for $1\leqslant v_i\leqslant k$, which are the elements which appear in the basic products. Suppose in the basic product w_s, i_p occurs l_p times, $l_p\geqslant 1$. Then the **height** of the basic product is $\sum l_p(r_p-1)+1$ and the **length** is $\sum l_p$. Clearly if w_s has height h_s, then $w_s\in\pi_{h_s}(W)$.

Then there is **Hilton's Theorem** [10].

Theorem 3.1. *Let the ordered basic products of $W = S_1 \vee \cdots \vee S_k$ be $w_1, w_2, \ldots, w_s, \ldots$ with the height of $w_s = h_s$. Then for every m,*

$$\pi_m(W) \approx \bigoplus_{s=1}^{\infty} \pi_m\big(S^{h_s}\big).$$

The isomorphism $\theta : \bigoplus_{s=1}^{\infty} \pi_m(S^{h_s}) \to \pi_m(W)$ is defined by

$$\theta \mid \pi_m\big(S^{h_s}\big) = w_{s*} : \pi_m\big(S^{h_s}\big) \to \pi_m(W).$$

The direct sum is finite for each m since $h_s \to \infty$.

Let $W = S^k \vee S^k$ in Theorem 3.1 and let h_s be the height of the sth basic product $w_s \in \pi_{h_s}(S^k \vee S^k)$. Furthermore, let φ be the standard comultiplication on S^k and let $pr_s : \bigoplus_{s=1}^{\infty} \pi_m(S^{h_s}) \to \pi_m(S^{h_s})$ be the projection onto the sth summand for $s = 1, 2, 3, \ldots$.

Define the **Hopf–Hilton invariant** $H_t : \pi_m(S^k) \to \pi_m(S^{h_{t+3}})$ by the composition

$$H_t = pr_{t+3}\theta^{-1}\varphi_*, \quad t = 0, 1, 2, \ldots$$

as in [10]. We rewrite these invariants as follows: Set

$$H_1^1 = H_0 : \pi_m\big(S^k\big) \to \pi_m\big(S^{2k-1}\big),$$
$$H_1^2 = H_1, \qquad H_2^2 = H_2 : \pi_m\big(S^k\big) \to \pi_m\big(S^{3k-2}\big)$$

and so on. Thus we have Hopf–Hilton invariants

$$H_1^l, H_2^l, \ldots, H_{t_l}^l : \pi_m\big(S^k\big) \to \pi_m\big(S^{(l+1)k-l}\big),$$

for $l = 1, 2, 3, \ldots$, where t_l is the number of basic products of length l.

If Y is any space and $\beta, \gamma \in \pi_r(Y)$ and $\alpha \in \pi_m(S^r)$, then there is **Hilton's Formula** [10]:

$$(\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha + [\beta, \gamma]H_1^1(\alpha) + \big[\beta, [\beta, \gamma]\big]H_1^2(\alpha) + \big[\gamma, [\beta, \gamma]\big]H_2^2(\alpha) + \cdots.$$

Corollary 3.2. *Let Y be any space and let $\beta, \gamma \in \pi_r(Y)$ and $\alpha \in \pi_m(S^r)$. If $H_j^i(\alpha) = 0$, for $i = 1, 2, \ldots, l-1$ and $j = 1, 2, \ldots, t_i$, and $m \leqslant (l+1)(r-1)$, then*

$$(\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha.$$

For the remainder of the paper we will use the following notation with $X = S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_t}$ and $\varphi_0 : X \to X \vee X$ the suspension comultiplication obtained from the suspension structure of X. Unless otherwise stated we will assume that the group structure on any homotopy set $[X, Y]$ is obtained from the cogroup structure (X, φ_0).

- $q_j : X \to S^{n_j}$ is the jth projection for $j = 1, 2, \ldots, t$.
- $i_j, i'_j : S^{n_j} \to S^{n_j} \vee S^{n_j}$ are the first and second inclusions, respectively, for $j = 1, 2, \ldots, t$.
- $\iota_1, \iota_2 : X \to X \vee X$ are the first and second inclusions, respectively.
- $\pi_1, \pi_2 : X \vee X \to X$ are first and second projections, respectively.
- $I_1, I_2, I_3 : X \to X \vee X \vee X$ are the first, second and third inclusions, respectively.
- $J_{12}, J_{23} : X \vee X \to X \vee X \vee X$ are the maps defined by $J_{12}(z) = (z, *)$ and $J_{23}(z) = (*, z)$, where $z \in X \vee X$.
- As before, $T : X \vee X \to X \vee X$ is the switching map and $\nabla : X \vee X \to X$ is the folding map.

Lemma 3.3. *(Cf. [3, Prop. 2.4].) Let $\varphi : X \to X \vee X$ be any map. Then φ is a comultiplication on X if and only if for all $j = 1, 2, \ldots, t$,*

$$\varphi k_j = \iota_1 k_j + \iota_2 k_j + P_j,$$

where $P_j \in \pi_{n_j}(X \vee X)$ has the property $\pi_1 P_j = 0 = \pi_2 P_j$ for each $j = 1, 2, \ldots, t$.

Proof. Set $P_j = \varphi k_j - \iota_1 k_j - \iota_2 k_j$. Clearly $\pi_1 P_j = 0 = \pi_2 P_j$. □

Furthermore, P_j is unique and is the sum of elements of the form $w_s\alpha_s$, where $\alpha_s \in \pi_{n_j}(S^{h_s})$ and $w_s \in \pi_{h_s}(X \vee X)$ are basic products of length $\geqslant 2$ in $X \vee X$ and height h_s of elements (written in order)

$$(k_1 \vee k_1)i_1, (k_2 \vee k_2)i_2, \ldots, (k_{j-1} \vee k_{j-1})i_{j-1}$$

and

$$(k_1 \vee k_1)i'_1, (k_2 \vee k_2)i'_2, \ldots, (k_{j-1} \vee k_{j-1})i'_{j-1},$$

such that each basic product contains at least one primed and one unprimed element. We observe that these $2j-2$ elements are

$$\iota_1k_1, \iota_1k_2, \dots, \iota_1k_{t-1}, \iota_2k_1, \iota_2k_2, \dots, \iota_2k_{t-1}.$$

Since the w_s are basic products, the elements α_s are uniquely determined by φ by Hilton's Theorem.

Definition 3.4. The element $P_j \in \pi_{n_j}(X \vee X)$ is called the **jth perturbation** of φ. We call $P = (P_1, P_2, \dots, P_t)$ the **perturbation** of φ and write $\varphi = \varphi_P = \varphi_{(P_1,P_2,\dots,P_t)}$. We call P or φ **one-stage** if there is some r with $1 \leqslant r \leqslant t$ such that $P_i = 0$ for all $i \neq r$.

Given two perturbations $P = (P_1, P_2, \dots, P_t)$ and $Q = (Q_1, Q_2, \dots, Q_t)$, we can add them in the obvious way

$$P + Q = (P_1 + Q_1, P_2 + Q_2, \dots, P_t + Q_t)$$

and take the negative of each

$$-P = (-P_1, -P_2, \dots, -P_t).$$

This allows us to add and subtract comultiplications on X. If φ and ψ are two comultiplications of X, then $\varphi = \varphi_P$ and $\psi = \varphi_Q$, for some perturbations P and Q. We define

$$\varphi + \psi = \varphi_{P+Q} \quad \text{and} \quad -\varphi = \varphi_{-P}.$$

This gives the set of comultiplications $\mathcal{C}(X)$ abelian group structure with unit φ_0. It is easily seen that this group is the same as the group defined in Proposition 2.1.

We next introduce some terminology that we will use in the rest of the paper.

Definition 3.5. Let $\varphi = \varphi_P$ be a comultiplication on X with perturbation P. Then P or φ is said to be **quadratic** if each P_j has only basic Whitehead products of length $\leqslant 2$, is said to be **strictly cubic** if each P_j has only basic Whitehead products of length $= 3$ and is said to be **cubic** if each P_j has only basic Whitehead products of length $\leqslant 3$.

4. Associativity

Lemma 4.1. *The comultiplication $\varphi = \varphi_P$, where $P = (P_1, P_2, \dots, P_t)$, is associative if and only if*

$$J_{12}P_r + (\varphi \vee 1)P_r = J_{23}P_r + (1 \vee \varphi)P_r$$

for all $r = 1, 2, \dots, t$.

Proof. The proof is straightforward. For details in the case of a wedge of two spheres, see [3, Proposition 2.6]. □

Now we consider a one-stage quadratic comultiplication $\varphi = \varphi_P$, where $P = (0, 0, \dots, 0, P_r, 0, \dots, 0)$, so that P_r consists of basic Whitehead products of length $\leqslant 2$. Thus P_r is a sum of elements of the form

$$P_r = [\iota_1k_i, \iota_2k_j]a_{ij},$$

for some $a_{ij} \in \pi_{n_r}(S^{h_{ij}})$, where $h_{ij} = n_i + n_j - 1$ is the height of the Whitehead product $[\iota_1k_i, \iota_2k_j]$ and $i, j = 1, 2, \dots, r-1$. Note that $a_{ij} = 0$ if $n_r < n_i + n_j - 1$.

Proposition 4.2. *With the notation above, assume that $n_r \leqslant 3n_i + 3n_j - 6$ and that the first Hopf–Hilton invariants $H_1^1(a_{ij})$ are trivial for each $i, j = 1, 2, \dots, r-1$. Then $\varphi = \varphi_P$ is associative.*

Proof. We will show that P_r satisfies Lemma 4.1. For this it suffices to show that

$$J_{12}\big([\iota_1k_i, \iota_2k_j]a_{ij}\big) + (\varphi \vee 1)\big([\iota_1k_i, \iota_2k_j]a_{ij}\big) = J_{23}\big([\iota_1k_i, \iota_2k_j]a_{ij}\big) + (1 \vee \varphi)\big([\iota_1k_i, \iota_2k_j]a_{ij}\big) \tag{4.1}$$

for all $i, j = 1, 2, \dots, r-1$. We have

$$J_{12}\big([\iota_1k_i, \iota_2k_j]a_{ij}\big) = [I_1k_i, I_2k_j]a_{ij}$$

and

$$J_{23}\big([\iota_1k_i, \iota_2k_j]a_{ij}\big) = [I_2k_i, I_3k_j]a_{ij}.$$

In addition,

$$\begin{aligned}(\varphi\vee 1)\big([\iota_1k_i,\iota_2k_j]a_{ij}\big) &= [J_{12}\varphi k_i, I_3k_j]a_{ij}\\ &= [I_1k_i+I_2k_i, I_3k_j]a_{ij}\\ &= \big([I_1k_i,I_3k_j]+[I_2k_i,I_3k_j]\big)a_{ij}\\ &= [I_1k_i,I_3k_j]a_{ij}+[I_2k_i,I_3k_j]a_{ij},\end{aligned}$$

by the Corollary 3.2, since all Hopf–Hilton invariants of a_{ij} are trivial by our hypotheses. Similarly,

$$(1\vee\varphi)\big([\iota_1k_i,\iota_2k_j]a_{ij}\big) = [I_1k_i,I_2k_j]a_{ij}+[I_1k_i,I_3k_j]a_{ij}.$$

This proves (4.1) and thus completes the proof. □

Remark 4.3. The inequality $n_r \leqslant 3n_i+3n_j-6$ and the vanishing of the first Hopf–Hilton invariants $H_1^1(a_{ij})$ were needed as hypotheses in the previous proposition only to show that, by Hilton's Formula,

$$\big([I_1k_i,I_3k_j]+[I_2k_i,I_3k_j]\big)a_{ij} = [I_1k_i,I_3k_j]a_{ij}+[I_2k_i,I_3k_j]a_{ij}$$

and similarly for $([I_1k_i,I_2k_j]+[I_1k_i,I_3k_j])a_{ij}$. It is therefore possible to weaken the inequality (that is, give a larger upper bound for n_r) and require more Hopf–Hilton invariants of the a_{ij} to be zero and still conclude that φ_P is associative. The statement of this result is somewhat cumbersome, and we do not give it.

We give two examples which bear on Proposition 4.2. The first shows that the condition on the Hopf–Hilton invariants in Proposition 4.2 cannot in general be deleted. The second shows that the conclusion of Proposition 4.2 is not true for arbitrary quadratic comultiplications.

Example 4.4.

1. We consider a one-stage quadratic comultiplication on a wedge X of three spheres such that

   $$\varphi k_3 = \iota_1k_3+\iota_2k_3+[\iota_1k_1,\iota_2k_2]a$$

   with $H_1^1(a)\neq 0$ and show that φ is not associative. For definiteness, take $X=S^2\vee S^3\vee S^7$ and let $a\in\pi_7(S^4)$ be an element of (classical) Hopf invariant 1. If we assume that φ is associative, then the calculation given in the proof of Proposition 4.2 shows that

   $$\big[[I_1k_1,I_3k_2],[I_2k_1,I_3k_2]\big] = \big[[I_1k_1,I_2k_2],[I_1k_1,I_3k_2]\big],$$

   since $H_1^1(a)=1$. But these are both basic products in $\pi_7(X\vee X\vee X)$ and so they cannot be equal.
2. We assume that $X=S^n\vee S^m\vee S^p$ and that φ is a comultiplication on X such that

   $$\varphi k_1 = \iota_1k_1+\iota_2k_1,$$
   $$\varphi k_2 = \iota_1k_2+\iota_2k_2+[\iota_1k_1,\iota_2k_1]a,$$
   $$\varphi k_3 = \iota_1k_3+\iota_2k_3+[\iota_1k_1,\iota_2k_2]b,$$

   where we assume all Hopf–Hilton invariants of b are zero. Then

   $$J_{12}P_3 = [I_1k_1,I_2k_2]b$$

   and with the vanishing of the Hopf–Hilton invariants,

   $$(\varphi\vee 1)P_3 = [I_1k_1,I_3k_2]b+[I_2k_1,I_3k_2]b.$$

   Similarly,

   $$J_{23}P_3 = [I_2k_1,I_3k_2]b$$

   and

   $$(1\vee\varphi)P_3 = [I_1k_1,I_2k_2]b+[I_1k_1,I_3k_2]b+\big[I_1k_1,[I_2k_1,I_3k_1]a\big]b.$$

   We assume that φ is associative, apply the equation in Lemma 4.1, cancel terms and obtain

   $$\big[I_1k_1,[I_2k_1,I_3k_1]a\big]b=0.$$

   But by taking $m=2n-1$, $p=3n-2$, $a=1$ and $b=1$, we see that

   $$\big[I_1k_1,[I_2k_1,I_3k_1]\big] = -(-1)^{n^2}\big[I_2k_1,[I_1k_1,I_3k_1]\big]-\big[I_3k_1,[I_1k_1,I_2k_1]\big]\neq 0$$

   since they are basic products in $\pi_{3n-2}(X\vee X\vee X)$. Thus φ is not associative.

We next consider associativity for one-stage strictly cubic comultiplications.

Definition 4.5. Let $\varphi = \varphi_P : X \to X \vee X$ be defined by

$$\begin{cases} \varphi k_1 = \iota_1 k_1 + \iota_2 k_1 \\ \quad\vdots \\ \varphi k_r = \iota_1 k_r + \iota_2 k_r + P_r \\ \quad\vdots \\ \varphi k_t = \iota_1 k_t + \iota_2 k_t. \end{cases}$$

Here $P = (0, \ldots, 0, P_r, 0, \ldots, 0)$ and

$$P_r = \sum_{a_1,b_1,c_1;\ a_1<b_1} \big[\iota_2 k_{c_1}, [\iota_1 k_{a_1}, \iota_1 k_{b_1}]\big]\alpha^r_{a_1b_1c_1} + \sum_{a_2,b_2,c_2;\ a_2\leqslant c_2} \big[\iota_1 k_{c_2}, [\iota_1 k_{a_2}, \iota_2 k_{b_2}]\big]\beta^r_{a_2b_2c_2} + \sum_{a_3,b_3,c_3} \big[\iota_2 k_{c_3}, [\iota_1 k_{a_3}, \iota_2 k_{b_3}]\big]\gamma^r_{a_3b_3c_3},$$

where $a_h, b_h, c_h = 1, 2, \ldots, r-1$ for $h = 1, 2, 3$ and the element $\alpha^r_{a_1b_1c_1} \in \pi_{n_r}(S^{n_{a_1}+n_{b_1}+n_{c_1}-2})$ and similarly for $\beta^r_{a_2b_2c_2}$ and $\gamma^r_{a_3b_3c_3}$. Furthermore, we assume the ordering $\iota_1 k_1, \iota_1 k_2, \ldots, \iota_1 k_{r-1}, \iota_2 k_1, \iota_2 k_2, \ldots, \iota_2 k_{r-1}$, so that the preceding Whitehead products are basic products. Indeed, these are all the Whitehead products that can appear in P_r and so the equality above for P_r yields the general expression for a one-stage strictly cubic comultiplication.

To simplify notation, we set $a_1, b_1, c_1 = i, j, l$, $a_2, b_2, c_2 = m, o, p$, and $a_3, b_3, c_3 = u, v, w$, respectively. We also put $\alpha = \alpha^r_{ijl}$, $\beta = \beta^r_{mop}$, $\gamma = \gamma^r_{uvw}$ and $\epsilon = n_m n_p - n_m - n_p$, $\delta = n_m n_o - n_o + n_o n_p - n_p$ and $\lambda = n_i n_j$. We consider terms of P_r consisting of one term from each summand, namely,

$$\begin{aligned} Q &= P_r(i, j, l, m, o, p, u, v, w) \\ &= \big[\iota_2 k_l, [\iota_1 k_i, \iota_1 k_j]\big]\alpha + \big[\iota_1 k_p, [\iota_1 k_m, \iota_2 k_o]\big]\beta + \big[\iota_2 k_w, [\iota_1 k_u, \iota_2 k_v]\big]\gamma, \end{aligned}$$

where $i < j$ and $m \leqslant p$.

We investigate the associativity of φ_P and state our result after giving the proof and introducing some notation. We assume in the proof that $n_r \leqslant 3n_{a_j} + 3n_{b_j} + 3n_{c_j} - 9$, $j = 1, 2, 3$ and that the first Hopf–Hilton invariants $H_1^1(\alpha^r_{ijl})$, $H_1^1(\beta^r_{mop})$ and $H_1^1(\gamma^r_{uvw})$ are trivial. With Q defined as above, we determine necessary and sufficient conditions for

$$J_{12}Q + (\varphi \vee 1)Q = J_{23}Q + (1 \vee \varphi)Q \tag{4.2}$$

to hold.

We compute $J_{12}Q$ and $J_{23}Q$.

$$\begin{aligned} J_{12}Q &= \big[I_2 k_l, [I_1 k_i, I_1 k_j]\big]\alpha + \big[I_1 k_p, [I_1 k_m, I_2 k_o]\big]\beta + \big[I_2 k_w, [I_1 k_u, I_2 k_v]\big]\gamma \quad \text{and} \\ J_{23}Q &= \big[I_3 k_l, [I_2 k_i, I_2 k_j]\big]\alpha + \big[I_2 k_p, [I_2 k_m, I_3 k_o]\big]\beta + \big[I_3 k_w, [I_2 k_u, I_3 k_v]\big]\gamma. \end{aligned}$$

We now compute $(\varphi \vee 1)Q$. Since $J_{12}\varphi k_s = J_{12}(\iota_1 k_s + \iota_2 k_s) = I_1 k_s + I_2 k_s$, we have

$$\begin{aligned} (\varphi \vee 1)Q &= \big[I_3 k_l, [J_{12}\varphi k_i, J_{12}\varphi k_j]\big]\alpha + \big[J_{12}\varphi k_p, [J_{12}\varphi k_m, I_3 k_o]\big]\beta + \big[I_3 k_w, [J_{12}\varphi k_u, I_3 k_v]\big]\gamma \\ &= \big[I_3 k_l, [I_1 k_i + I_2 k_i, I_1 k_j + I_2 k_j]\big]\alpha + \big[I_1 k_p + I_2 k_p, [I_1 k_m + I_2 k_m, I_3 k_o]\big]\beta + \big[I_3 k_w, [I_1 k_u + I_2 k_u, I_3 k_v]\big]\gamma \\ &= \big(\big[I_3 k_l, [I_1 k_i, I_1 k_j]\big] + \big[I_3 k_l, [I_1 k_i, I_2 k_j]\big] + (-1)^\lambda \big[I_3 k_l, [I_1 k_j, I_2 k_i]\big] + \big[I_3 k_l, [I_2 k_i, I_2 k_j]\big]\big)\alpha \\ &\quad + \big(\big[I_1 k_p, [I_1 k_m, I_3 k_o]\big] + \big[I_1 k_p, [I_2 k_m, I_3 k_o]\big] + \big[I_2 k_p, [I_1 k_m, I_3 k_o]\big] + \big[I_2 k_p, [I_2 k_m, I_3 k_o]\big]\big)\beta \\ &\quad + \big(\big[I_3 k_w, [I_1 k_u, I_3 k_v]\big] + \big[I_3 k_w, [I_2 k_u, I_3 k_v]\big]\big)\gamma, \end{aligned}$$

and α, β and γ go inside the parentheses, by Corollary 3.2. Similarly,

$$\begin{aligned} (1 \vee \varphi)Q &= \big[J_{23}\varphi k_l, [I_1 k_i, I_1 k_j]\big]\alpha + \big[I_1 k_p, [I_1 k_m, J_{23}\varphi k_o]\big]\beta + \big[J_{23}\varphi k_w, [I_1 k_u, J_{23}\varphi k_v]\big]\gamma \\ &= \big[I_2 k_l + I_3 k_l, [I_1 k_i, I_1 k_j]\big]\alpha + \big[I_1 k_p, [I_1 k_m, I_2 k_o + I_3 k_o]\big]\beta + \big[I_2 k_w + I_3 k_w, [I_1 k_u, I_2 k_v + I_3 k_v]\big]\gamma \\ &= \big[I_2 k_l, [I_1 k_i, I_1 k_j]\big]\alpha + \big[I_3 k_l, [I_1 k_i, I_1 k_j]\big]\alpha + \big[I_1 k_p, [I_1 k_m, I_2 k_o]\big]\beta + \big[I_1 k_p, [I_1 k_m, I_3 k_o]\big]\beta \\ &\quad + \big[I_2 k_w, [I_1 k_u, I_2 k_v]\big]\gamma + \big[I_2 k_w, [I_1 k_u, I_3 k_v]\big]\gamma + \big[I_3 k_w, [I_1 k_u, I_2 k_v]\big]\gamma + \big[I_3 k_w, [I_1 k_u, I_3 k_v]\big]\gamma. \end{aligned}$$

Thus we see that (4.2) holds if and only if

$$0 = \big[I_3k_l, [I_1k_i, I_2k_j]\big]\alpha + (-1)^{\lambda}\big[I_3k_l, [I_1k_j, I_2k_i]\big]\alpha + \big[I_1k_p, [I_2k_m, I_3k_o]\big]\beta + \big[I_2k_p, [I_1k_m, I_3k_o]\big]\beta$$
$$- \big[I_2k_w, [I_1k_u, I_3k_v]\big]\gamma - \big[I_3k_w, [I_1k_u, I_2k_v]\big]\gamma.$$

But $[I_1k_p, [I_2k_m, I_3k_o]]$ is not a basic product, so, using the Jacobi identity, we write it as a sum of basic products as follows:

$$\big[I_1k_p, [I_2k_m, I_3k_o]\big] = -(-1)^{n_mn_p-n_m-n_p}\big[I_2k_m, [I_1k_p, I_3k_o]\big] - (-1)^{n_mn_o-n_o+n_on_p-n_p}\big[I_3k_o, [I_1k_p, I_2k_m]\big].$$

Thus (4.2) holds if and only if

$$0 = \big[I_3k_l, [I_1k_i, I_2k_j]\big]\alpha + (-1)^{\lambda}\big[I_3k_l, [I_1k_j, I_2k_i]\big]\alpha - (-1)^{\delta}\big[I_3k_o, [I_1k_p, I_2k_m]\big]\beta - \big[I_3k_w, [I_1k_u, I_2k_v]\big]\gamma$$
$$- (-1)^{\epsilon}\big[I_2k_m, [I_1k_p, I_3k_o]\big]\beta + \big[I_2k_p, [I_1k_m, I_3k_o]\big]\beta - \big[I_2k_w, [I_1k_u, I_3k_v]\big]\gamma, \tag{4.3}$$

where $\epsilon = n_mn_p - n_m - n_p$, $\delta = n_mn_o - n_o + n_on_p - n_p$ and $\lambda = n_in_j$.

To simplify matters, we fix the following notation:

- $a = [I_3k_l, [I_1k_i, I_2k_j]]$.
- $b = [I_3k_l, [I_1k_j, I_2k_i]]$.
- $c = [I_3k_o, [I_1k_p, I_2k_m]]$.
- $d = [I_3k_w, [I_1k_u, I_2k_v]]$.
- $e = [I_2k_m, [I_1k_p, I_3k_o]]$.
- $f = [I_2k_p, [I_1k_m, I_3k_o]]$.
- $g = [I_2k_w, [I_1k_u, I_3k_v]]$.

Note that the above terms are all basic products and that $a \neq b$ and $a \neq c$ because $i < j$ and $m \leqslant p$. Thus there are seven possibilities for the first four terms of (4.3), depending on whether or not some of the Whitehead products are equal. We write these seven cases as

$$(a, b, c, d),\ (a, b, c, b),\ (a, b, c, c),\ (a, b, b, b),\ (a, b, c, a),\ (a, b, b, d),\ (a, b, b, a).$$

For the last three terms of (4.3) we have the five cases

$$(e, f, g),\ (e, f, f),\ (e, e, g),\ (e, e, e),\ (e, f, e).$$

Here, the same letters means the same basic products and different letters means different basic products (e.g., (a, b, c, b) means $b = d$ and $a \neq b$, $a \neq c$ and $b \neq c$).

We compute case by case. We note that (4.3) holds if and only if

$$0 = \big(a\alpha + (-1)^{\lambda}b\alpha + (-1)^{\delta+1}c\beta - d\gamma\big) + \big((-1)^{\epsilon+1}e\beta + f\beta - g\gamma\big). \tag{4.4}$$

First, by using Hilton's Theorem, we compute T_1, the sum of the first four terms in (4.4).

Case 1: (a, b, c, d). Since a, b, c and d are all basic products, $T_1 = 0$ if and only if $\alpha = \beta = \gamma = 0$.
Case 2: (a, b, c, b). $T_1 = 0$ if and only if $\alpha = \beta = \gamma = 0$.
Case 3: (a, b, c, c). $T_1 = 0$ if and only if $\alpha = 0$ and $\gamma = (-1)^{\delta+1}\beta$.
Case 4: (a, b, b, b). $T_1 = 0$ if and only if $\alpha = 0$ and $\gamma = (-1)^{\delta+1}\beta$.
Case 5: (a, b, c, a). $T_1 = 0$ if and only if $\alpha = \beta = \gamma = 0$.
Case 6: (a, b, b, d). $T_1 = 0$ if and only if $\alpha = \beta = \gamma = 0$.
Case 7: (a, b, b, a). $T_1 = 0$ if and only if $\alpha = \gamma$, $(-1)^{\lambda}\alpha = (-1)^{\delta}\beta$.

Secondly, we determine T_2, the sum of the last three terms in (4.4).

Case 8: (e, f, g). $T_2 = 0$ if and only if $\beta = \gamma = 0$.
Case 9: (e, f, f). $T_2 = 0$ if and only if $\beta = \gamma = 0$.
Case 10: (e, e, g). $T_2 = 0$ if and only if $(-1)^{\epsilon+1}\beta + \beta = 0$, $\gamma = 0$.
Case 11: (e, e, e). $T_2 = 0$ if and only if $\gamma = (-1)^{\epsilon+1}\beta + \beta$.
Case 12: (e, f, e). $T_2 = 0$ if and only if $\beta = \gamma = 0$.

We now give conditions for the comultiplication φ_P to be associative. By Lemma 4.1, for φ_P to be associative it is necessary and sufficient that (4.2) holds for every Q, where $Q = P_r(i, j, l, m, o, p, u, v, w)$, that is, that (4.4) holds for every Q. The term T_1 of (4.4) is any one of cases 1–7 and the term T_2 is any one of cases 8–12. For (4.4) to hold it is necessary and sufficient that both $T_1 = 0$ and $T_2 = 0$. Clearly, when $\alpha = \beta = \gamma = 0$, the comultiplication φ_P is associative. We leave it as an exercise for the reader to show that the only other case when φ_P is associative is the following:

Case 3 and Case 11: ϵ is odd, δ is even, $\alpha = 0$, $\gamma = -\beta$ and $3\beta = 0$.

We now state the proposition on strictly cubic one-stage comultiplications which we have just proved. We adopt the notation of the preceding proof.

Proposition 4.6. *Assume that $\varphi = \varphi_P$ is given by Definition 4.5 with $n_r \leqslant 3n_{a_j} + 3n_{b_j} + 3n_{c_j} - 9$, $j = 1, 2, 3$ and that the first Hopf–Hilton invariants $H_1^1(\alpha^r_{a_1b_1c_1})$, $H_1^1(\beta^r_{a_2b_2c_2})$ and $H_1^1(\gamma^r_{a_3b_3c_3})$ are trivial. Then φ is associative if and only if*

1. *$P_r = \sum_q [\iota_1 k_q, [\iota_1 k_q, \iota_2 k_q]]\beta_q - \sum_q [\iota_2 k_q, [\iota_1 k_q, \iota_2 k_q]]\beta_q$ for $k_q : S^{n_q} \to X = S^{n_1} \vee \cdots \vee S^{n_t}$ inclusion into the qth summand and $\beta_q \in \pi_{n_r}(S^{3n_q-2})$ with $3\beta_q = 0$, where n_q is odd, or*
2. *All $\alpha^r_{a_1b_1c_1} = \beta^r_{a_2b_2c_2} = \gamma^r_{a_3b_3c_3} = 0$, that is, $\varphi = \iota_1 k_s + \iota_2 k_s$ for all s, otherwise.*

Remark 4.7.

1. In analogy to Remark 4.3, it is possible in Proposition 4.6 to weaken the inequality (that is, give a larger upper bound for n_r) and require more Hopf–Hilton invariants of $\alpha^r_{a_1b_1c_1}$, $\beta^r_{a_2b_2c_2}$ and $\gamma^r_{a_3b_3c_3}$ to be zero, and thus derive necessary and sufficient conditions for φ to be associative.
2. By combining Propositions 4.2 and 4.6, we obtain an associativity result for arbitrary one-stage cubic comultiplications.

We conclude the section with two examples to illustrate this Proposition 4.6.

Example 4.8.

1. Let $X = S^3 \vee \cdots \vee S^{10} \vee \cdots \vee S^{n_t}$, where $n_r = 10$. Define a one-stage cubic comultiplication $\varphi = \varphi_P : X \to X \vee X$ by
$$P_r = \big[\iota_1 k_1, [\iota_1 k_1, \iota_2 k_1]\big]\beta - \big[\iota_2 k_1, [\iota_1 k_1, \iota_2 k_1]\big]\beta,$$
where $P = (0, \dots, 0, P_r, 0, \dots, 0)$, $k_1 : S^3 \to X$ is the inclusion map and β denotes an element of order 3 in the group $\pi_{10}(S^7) \cong \mathbb{Z}_{24}$ [14, p. 186]. Then the hypotheses of Proposition 4.6 are satisfied, so φ is associative.
2. Let $X = S^3 \vee S^5 \vee \cdots \vee S^{24} \vee \cdots \vee S^{n_t}$, with $n_r = 24$. We define a one-stage cubic comultiplication $\varphi = \varphi_P : X \to X \vee X$ by
$$P_r = \big[\iota_1 k_2, [\iota_1 k_2, \iota_2 k_2]\big]\beta - \big[\iota_2 k_2, [\iota_1 k_2, \iota_2 k_2]\big]\beta,$$
where $k_2 : S^5 \to X$ is the inclusion map and β is an element of order 3 in the group $\pi_{24}(S^{13}) \cong \mathbb{Z}_{504}$ [14, p. 188]. Then the hypotheses of Proposition 4.6 are satisfied, so φ is associative.

5. Commutativity

In this section we obtain results on the commutativity of quadratic and cubic comultiplications. We first consider quadratic comultiplications $\varphi = \varphi_P = \varphi_{(P_1, P_2, \dots, P_t)}$ on X. Then for each $r = 2, 3, \dots, t$,
$$P_r = \sum_{i,j=1}^{r-1} [\iota_1 k_i, \iota_2 k_j] a^r_{ij},$$
for some $a^r_{ij} \in \pi_{n_r}(S^{n_i+n_j-1})$.

Proposition 5.1. *Let $\varphi = \varphi_P$ be a quadratic comultiplication as above with $H_1^1(a^r_{ij}) = 0$ if n_i and n_j are odd. Then φ is commutative if and only if* (1) *$a^r_{ij} = a^r_{ji}$, whenever n_i or n_j is even or* (2) *$a^r_{ij} = -a^r_{ji}$, whenever n_i and n_j are odd.*

Proof. We determine TP_r since $TP_r = P_r$ is a necessary and sufficient condition for φ to be commutative. We have
$$\begin{aligned} TP_r &= \sum_{i,j=1}^{r-1} [\iota_2 k_i, \iota_1 k_j] a^r_{ij} \\ &= \sum_{i,j=1}^{r-1} \big((-1)^{n_i n_j} [\iota_1 k_j, \iota_2 k_i]\big) a^r_{ij} \\ &= \sum_{i,j=1}^{r-1} \big((-1)^{n_i n_j} [\iota_1 k_i, \iota_2 k_j]\big) a^r_{ji}. \end{aligned}$$
If n_i or n_j is even, then $(-1)^{n_i n_j}[\iota_1 k_i, \iota_2 k_j] a^r_{ji} = [\iota_1 k_i, \iota_2 k_j] a^r_{ji}$ and $TP_r = P_r$. If n_i and n_j are odd, then $(-1)^{n_i n_j}[\iota_1 k_i, \iota_2 k_j] a^r_{ji} = [\iota_1 k_i, \iota_2 k_j](-1) a^r_{ji}$, where -1 is the map of degree -1 of $S^{n_i+n_j-1}$. By [10, p. 167], $(-1)a^r_{ji} = -a^r_{ji} + [1, 1]H_1^1(a^r_{ji}) = -a^r_{ji}$.

Thus $(-1)^{n_i n_j}[\iota_1 k_i, \iota_2 k_j]a^r_{ji} = [\iota_1 k_i, \iota_2 k_j](-a^r_{ji})$ in this case. Therefore if the hypotheses of (1) or (2) holds, $TP_r = P_r$. Conversely, if $TP_r = P_r$, then $[\iota_1 k_i, \iota_2 k_j]a^r_{ij} = [\iota_1 k_i, \iota_2 k_j]a^r_{ji}$ if n_i or n_j is even, and $[\iota_1 k_u, \iota_2 k_v]a^r_{uv} = [\iota_1 k_u, \iota_2 k_v](-a^r_{vu})$ if n_u and n_v are odd. But $[\iota_1 k_u, \iota_2 k_v] = (k_u \vee k_v)[i_u, i'_v]$. Furthermore, $(k_u \vee k_v)$ and $[i_u, i'_v]$ both induce monomorphisms of homotopy groups and therefore so do $[\iota_1 k_u, \iota_2 k_v]$ and $[\iota_1 k_i, \iota_2 k_j]$. Therefore $a^r_{ij} = a^r_{ji}$ in (1) and $a^r_{ij} = -a^r_{ji}$ in (2). $\square$

Remark 5.2. From the proof we see that the condition $H^1_1(a^r_{ji}) = 0$ in the statement of Proposition 5.1 can be replaced with $[1,1]H^1_1(a^r_{ji}) = 0$.

We next consider commutativity of strict cubic comultiplications of X.

Proposition 5.3. *Let $\varphi = \varphi_P = \varphi_{(P_1, P_2, \ldots, P_t)}$ be a strict cubic comultiplication, which has the following form for each $r = 1, 2, \ldots, t$*

$$P_r = \sum_{a_2, b_2, c_2;\, a_2 \leqslant c_2} \big[\iota_1 k_{c_2}, [\iota_1 k_{a_2}, \iota_2 k_{b_2}]\big]\beta^r_{a_2 b_2 c_2} + \sum_{a_3, b_3, c_3;\, b_3 \leqslant c_3} \big[\iota_2 k_{c_3}, [\iota_1 k_{a_3}, \iota_2 k_{b_3}]\big]\gamma^r_{a_3 b_3 c_3},$$

where $a_h, b_h, c_h = 1, 2, \ldots, r-1$ for $h = 2, 3$ and the elements $\beta^r_{a_2 b_2 c_2} \in \pi_{n_r}(S^{n_{a_2}+n_{b_2}+n_{c_2}-2})$ and $\gamma^r_{a_3 b_3 c_3} \in \pi_{n_r}(S^{n_{a_3}+n_{b_3}+n_{c_3}-2})$. For all $u, v, w = 1, 2, \ldots, r-1$, we assume that (1) if n_u or n_v is even ($u \leqslant w$), then $\beta^r_{uvw} = \gamma^r_{vuw}$ or (2) if n_u and n_v are odd, then $\beta^r_{uvw} = -\gamma^r_{vuw}$ and $H^1_1(\beta^r_{uvw}) = 0$. Then φ is commutative.

Proof. It suffices to show that $P_r = TP_r$.

$$\begin{aligned}
TP_r &= \sum_{a_2, b_2, c_2;\, a_2 \leqslant c_2} \big[\iota_2 k_{c_2}, [\iota_2 k_{a_2}, \iota_1 k_{b_2}]\big]\beta^r_{a_2 b_2 c_2} + \sum_{a_3, b_3, c_3;\, b_3 \leqslant c_3} \big[\iota_1 k_{c_3}, [\iota_2 k_{a_3}, \iota_1 k_{b_3}]\big]\gamma^r_{a_3 b_3 c_3} \\
&= \sum_{a_2, b_2, c_2;\, a_2 \leqslant c_2} \big((-1)^{n_{a_2} n_{b_2}}\big[\iota_2 k_{c_2}, [\iota_1 k_{b_2}, \iota_2 k_{a_2}]\big]\big)\beta^r_{a_2 b_2 c_2} + \sum_{a_3, b_3, c_3;\, b_3 \leqslant c_3} \big((-1)^{n_{a_3} n_{b_3}}\big[\iota_1 k_{c_3}, [\iota_1 k_{b_3}, \iota_2 k_{a_3}]\big]\big)\gamma^r_{a_3 b_3 c_3} \\
&= \sum_{a_3, b_3, c_3;\, b_3 \leqslant c_3} \big((-1)^{n_{a_3} n_{b_3}}\big[\iota_2 k_{c_3}, [\iota_1 k_{a_3}, \iota_2 k_{b_3}]\big]\big)\beta^r_{b_3 a_3 c_3} + \sum_{a_2, b_2, c_2;\, a_2 \leqslant c_2} \big((-1)^{n_{a_2} n_{b_2}}\big[\iota_1 k_{c_2}, [\iota_1 k_{a_2}, \iota_2 k_{b_2}]\big]\big)\gamma^r_{b_2 a_2 c_2}.
\end{aligned}$$

Since $(-b)a = -(ba) + [b,b]H^1_1(a)$, it follows that $TP_r = P_r$ by (1) and (2). $\square$

Remark 5.4. It is possible to combine Propositions 5.1 and 5.3 and obtain commutative cubic comultiplications.

6. Algebraic loops

Let L be a set with binary operation '$+$'. Then L is called a **loop** if for every $a, b \in L$, the equations $a + x = b$ and $y + a = b$ have unique solutions x, y in L. All of the loops in this section will have a unit e, that is, $a + e = a = e + a$, for every $a \in L$, and we shall refer to them as **algebraic loops**. Let $L(a)$ and $R(a)$ be left and right inverses of $a \in L$ under the operation $+$ so that $L(a) + a = e = a + R(a)$. An algebraic loop L is said to be **inversive** if $L(a) = R(a)$ for all $a \in L$. An algebraic loop L is called **power-associative** if $(a + a) + a = a + (a + a)$, for all $a \in L$, and is called **Moufang** if $(a + b) + (c + a) = (a + (b + c)) + a$ for all $a, b, c \in L$ [6]. Clearly an associative binary operation is Moufang. We note that if L is a Moufang loop, then it is not difficult to show that L is inversive and power-associative.

The following result (see [11, Theorem 2.3] and [1, Proposition 1.13]) is the dual of a well-known theorem of James [12] for H-spaces and gives a connection between co-H-spaces and algebraic loops. We adopt the following notation: if (X, φ) is a co-H-space and W is any space, then the set $[X, W]$ with the binary operation induced by φ will be denoted $[X, W]_\varphi$.

Proposition 6.1. *If (X, φ) is a 1-connected co-H-space and W is any space, then the set $[X, W]_\varphi$ is an algebraic loop.*

Moreover, it is well known that $[X, W]_\varphi$ is also an algebraic loop if (X, φ) is a co-H-space and W is a nilpotent space [11].

For a comultiplication φ on any co-H-space X, we observe that φ is associative if and only if the binary operation in $[X, W]_\varphi$ is associative for all spaces W, and that φ is commutative if and only if the binary operation in $[X, W]_\varphi$ is commutative for all spaces W. This leads us to the following definition.

Definition 6.2. If (X, φ) is a co-H-space, then we call φ **inversive, power-associative or Moufang** if the algebraic loop $[X, W]_\varphi$ is inversive, power-associative or Moufang, respectively, for all spaces W.

It is possible to give equivalent intrinsic definitions of the notions in Definition 6.2, that is, definitions that only involve the comultiplication and not the homotopy set $[X, W]$.

It is natural to ask: for which comultiplications φ on a co-H-space X is φ inversive, power-associative or Moufang? In this section we answer this question for certain comultiplications on a wedge of spheres.

We let $X = S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_t}$ and let $\varphi : X \to X \vee X$ be a comultiplication. Then the composition

$$S^{n_j} \xrightarrow{k_j} X \xrightarrow{\varphi} X \vee X \xrightarrow{q_j \vee q_j} S^{n_j} \vee S^{n_j},$$

where k_j is the inclusion and q_j is the projection, is a comultiplication on S^{n_j} which we denote by $\varphi \mid S^{n_j}$. Note that $\varphi \mid S^{n_j}$ is the unique comultiplication on S^{n_j} obtained from the suspension structure of S^{n_j}, and that it is a commutative cogroup comultiplication. For any space W, we then have the following bijection of sets

$$\begin{aligned}[X, W]_\varphi &\approx \big[S^{n_1}, W\big]_{\varphi|S^{n_1}} \oplus \big[S^{n_2}, W\big]_{\varphi|S^{n_2}} \oplus \cdots \oplus \big[S^{n_t}, W\big]_{\varphi|S^{n^t}} \\ &= \pi_{n_1}(W) \oplus \pi_{n_2}(W) \oplus \cdots \oplus \pi_{n_t}(W).\end{aligned}$$

We identify the set $[X, W]_\varphi$ with the set which is the direct sum of homotopy groups, and write '$+_\varphi$' for the binary operation in $[X, W]_\varphi$ and '$+$' for the group operation in homotopy groups.

By Lemma 3.3,

$$\varphi k_r = \iota_1 k_r + \iota_2 k_r + P_r,$$

with $P_r \in \pi_{n_r}(X \vee X)$ the sum of elements of the form $w_s \alpha_s$, where $\alpha_s \in \pi_{n_r}(S^{h_s})$ and $w_s \in \pi_{h_s}(X \vee X)$ are basic products of length $\geqslant 2$ in $X \vee X$ and height h_s. In this section we investigate the loop properties in Definition 6.2 for quadratic and cubic comultiplications. We begin with the quadratic case. Assume the φ is a quadratic comultiplication and so for each k we have

$$P_k = \sum_{i,j=1}^{k-1} [\iota_1 k_i, \iota_2 k_j] a^k_{ij}.$$

Let $x, y \in [X, W]_\varphi$ and write $x = (x_1, x_2, \ldots, x_t)$ and $y = (y_1, y_2, \ldots, y_t)$, where $x_j, y_j \in \pi_{n_j}(W)$, for $j = 1, 2, \ldots, t$. With the notation above, the next result follows easily.

Lemma 6.3.

$$x +_\varphi y = (x_1, x_2, \ldots, x_t) +_\varphi (y_1, y_2, \ldots, y_t) = (z_1, z_2, \ldots, z_t), \quad \textit{where}$$
$$z_k = x_k + y_k + \sum_{i,j=1}^{k-1} [x_i, y_j] a^k_{ij},$$

for all $k = 1, 2, \ldots, t$.

Proposition 6.4. *Let φ be a one-stage quadratic comultiplication as defined above with $P_k = 0$ for $k \neq r$. Then*

1. *φ is inversive.*
2. *φ is power-associative.*
3. *φ is Moufang if $n_r \leqslant 3n_i + 3n_j - 6$ and the Hopf–Hilton invariants $H^1_1(a_{ij}) = 0$, for $i, j = 1, 2, \ldots, r - 1$.*

Proof. 1. Let $x = (x_1, x_2, \ldots, x_t) \in [X, W]_\varphi$ and $L(x) = \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$. Then

$$(\lambda +_\varphi x)_k = \begin{cases} \lambda_k + x_k & \text{if } k \neq r, \\ \lambda_r + x_r + \sum_{i,j=1}^{r-1} [\lambda_i, x_j] a^r_{ij} & \text{if } k = r. \end{cases}$$

Since $(\lambda +_\varphi x)_k = 0$, we have $\lambda_k = -x_k$ for $k \neq r$ and

$$\lambda_r = -x_r - \sum_{i,j=1}^{r-1} [\lambda_i, x_j] a^r_{ij}.$$

Similarly if $\rho = (\rho_1, \rho_2, \ldots, \rho_t) = R(x)$, then $\rho_k = -x_k$ for $k \neq r$ and

$$\rho_r = -x_r - \sum_{i,j=1}^{r-1} [x_i, \rho_j] a^r_{ij}.$$

Then for $i, j < r$,

$$[\lambda_i, x_j]a^r_{ij} = [-x_i, x_j]a^r_{ij} = \bigl(-[x_i, x_j]\bigr)a^r_{ij} \quad \text{and}$$
$$[x_i, \rho_j]a^r_{ij} = [x_i, -x_j]a^r_{ij} = \bigl(-[x_i, x_j]\bigr)a^r_{ij}.$$

Therefore $L(x) = R(x)$.

2. We prove that φ is power-associative. It is easily shown that

$$\bigl(x +_\varphi (x +_\varphi x)\bigr)_k = \begin{cases} 3x_k & \text{if } k \neq r, \\ 3x_r + \sum_{i,j=1}^{r-1} [x_i, x_j]a^r_{ij} + \sum_{i,j=1}^{r-1} [x_i, 2x_j]a^r_{ij} & \text{if } k = r \end{cases}$$

and

$$\bigl((x +_\varphi x) +_\varphi x\bigr)_k = \begin{cases} 3x_k & \text{if } k \neq r, \\ 3x_r + \sum_{i,j=1}^{r-1} [x_i, x_j]a^r_{ij} + \sum_{i,j=1}^{r-1} [2x_i, x_j]a^r_{ij} & \text{if } k = r. \end{cases}$$

But $[x_i, 2x_j]a^r_{ij} = (2[x_i, x_j])a^r_{ij} = [2x_i, x_j]a^r_{ij}$, and so φ is power-associative.

3. This is proved analogously to 1 and 2 (but see Remark 6.5 for another proof). $\square$

Remark 6.5. In Proposition 6.4, all of the conclusions hold whenever φ is associative since associativity implies the three loop properties listed in the proposition. But in Proposition 4.2, associativity is proved for a one-stage quadratic comultiplication φ with restrictions on n_r and the first Hopf–Hilton invariant. We now observe that we have proved that 1 and 2 of Proposition 6.4 hold without any restrictions on a one-stage quadratic comultiplication. However, the hypothesis of 3 implies that φ is associative by Proposition 4.2. Thus 3 follows from Proposition 4.2.

We next give an example to show that quadratic comultiplications which are not one-stage are not necessarily inversive or power-associative.

Example 6.6. Let $X = S^2 \vee S^3 \vee S^4 \vee S^5$ and define a quadratic comultiplication φ on X by $P_1 = P_2 = 0$, $P_3 = [\iota_1 k_1, \iota_2 k_2]$ and $P_4 = [\iota_1 k_1, \iota_2 k_3]$. We will use the notation of the proof of Proposition 6.4.

We first show that φ is not inversive. Let $L(x) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and let $R(x) = (\rho_1, \rho_2, \rho_3, \rho_4)$. Then $\lambda_i = -x_i = \rho_i$ for $i = 1, 2$ and $\lambda_3 = -x_3 + [x_1, x_2] = \rho_3$. Also

$$\begin{aligned} \lambda_4 &= -x_4 - [\lambda_1, x_3] = -x_4 + [x_1, x_3] \quad \text{and} \\ \rho_4 &= -x_4 - [x_1, \rho_3] \\ &= -x_4 - \bigl[x_1, -x_3 + [x_1, x_2]\bigr] \\ &= -x_4 + [x_1, x_3] - \bigl[x_1, [x_1, x_2]\bigr]. \end{aligned}$$

If we set $W = X \vee X$ and $x_1 = \iota_1 k_1$ and $x_2 = \iota_2 k_2$, then $[x_1, [x_1, x_2]] \neq 0$, since it is a basic product. Therefore in this case, $\lambda_4 \neq \rho_4$, and so $L(x) \neq R(x)$. Hence φ is not inversive.

Next we show that φ is not power-associative. Clearly $(x +_\varphi x)_4 = 2x_4 + [x_1, x_3]$. Furthermore,

$$\begin{aligned} \bigl(x +_\varphi (x +_\varphi x)\bigr)_4 &= 3x_4 + [x_1, x_3] + \bigl[x_1, (x +_\varphi x)_3\bigr] \\ &= 3x_4 + [x_1, x_3] + \bigl[x_1, 2x_3 + [x_1, x_2]\bigr] \\ &= 3x_4 + 3[x_1, x_3] + \bigl[x_1, [x_1, x_2]\bigr] \quad \text{and} \\ \bigl((x +_\varphi x) +_\varphi x\bigr)_4 &= 3x_4 + [x_1, x_3] + [2x_1, x_3] \\ &= 3x_4 + 3[x_1, x_3]. \end{aligned}$$

Since $[x_1, [x_1, x_2]] \neq 0$, for some W and some x_1 and x_2 as above, it follows that φ is not power-associative.

We note that $\varphi_{(0,0,P_3,0)} + \varphi_{(0,0,0,P_4)} = \varphi$, and so this example also shows that the sum of two associative comultiplications need not be associative.

We next obtain some loop properties for general one-stage comultiplications on X, not just for quadratic or cubic one-stage ones. Since the calculations for the Moufang property are very complicated, we restrict our attention to inversivity and power-associativity.

Let φ be a one-stage comultiplication on $X = S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_t}$ with perturbations $P_k = 0$ for $k \neq r$. We have seen in Section 3 that P_r is a sum of terms of the form $w_s \alpha_s$, where $\alpha_s \in \pi_{n_r}(S^{h_s})$ and $w_s \in \pi_{h_s}(X \vee X)$ are basic Whitehead products of length $\geqslant 2$ in $X \vee X$ and height h_s of elements (written in order) chosen from

$$\iota_1 k_1, \iota_1 k_2, \ldots, \iota_1 k_{t-1}, \iota_2 k_1, \iota_2 k_2, \ldots, \iota_2 k_{t-1}, \tag{6.1}$$

such that each basic product contains at least one of the first $t-1$ terms and at least one of the last $t-1$ terms.

Definition 6.7. For a basic product w_s as above, we define $\kappa_1(w_s)$ to be the number of terms from the first $t-1$ terms of (6.1) that w_s contains and define $\kappa_2(w_s)$ to be the number of terms from the last $t-1$ terms of (6.1) that w_s contains. In this definition we count repetitions.

Proposition 6.8. *Let φ be any one-stage comultiplication on X with $P_k = 0$ for $k \neq r$.*

1. *Suppose for each basic product w_s in P_r, we have*
$$\kappa_1(w_s) \equiv \kappa_2(w_s) \mod 2.$$
Then φ is inversive.
2. *Suppose for each basic product w_s in P_r, we have $\kappa_1(w_s) = \kappa_2(w_s)$. Then φ is power-associative.*

The proof should be clear by looking closely at the proofs of Propositions 6.4 and 6.9, and hence is omitted.

We conclude this section by studying the loop properties for a specific class of cubic comultiplications which are not covered by Proposition 6.8.

Let φ be a one-stage cubic comultiplication on X with $P_k = 0$ for $k \neq r$ and

$$P_r = \sum_{i,j=1}^{r-1} \big[\iota_2 k_i, [\iota_1 k_j, \iota_2 k_j]\big] c_{ij},$$

where $c_{ij} \in \pi_{n_r}(S^{n_i+2n_j-2})$. Let $x, y \in [X, W]_\varphi$ and set $x = (x_1, x_2, \dots, x_t)$ and $y = (y_1, y_2, \dots, y_t)$, where $x_j, y_j \in \pi_{n_j}(W)$, for $j = 1, 2, \dots, t$. Then

$$(x +_\varphi y)_k = \begin{cases} x_k + y_k & \text{if } k \neq r, \\ x_r + y_r + \sum_{i,j=1}^{r-1} [y_i, [x_j, y_j]] c_{ij} & \text{if } k = r. \end{cases}$$

For notational convenience, we write the triple Whitehead product $[a, [b, c]]$ as $[a, b, c]$.

Proposition 6.9. *Suppose that φ is the comultiplication defined above.*

1. *Then φ is inversive if and only if*
$$\big(-[x_i, x_j, x_j]\big) c_{ij} = [x_i, x_j, x_j] c_{ij},$$
for all spaces W, all $x \in [X, W]_\varphi$ and all $i, j < r$.
2. *Then φ is power-associative if and only if*
$$\big(4[x_i, x_j, x_j]\big) c_{ij} = \big(2[x_i, x_j, x_j]\big) c_{ij},$$
for all spaces W and all $x \in [X, W]_\varphi$.
3. *Assume that $H_1^1(c_{ij}) = 0$ and $n_r \leqslant 3n_i + 6n_j - 9$ for all $i, j < r$. Then φ is Moufang if and only if*
$$\begin{aligned} 0 = &\sum_{i,j=1}^{r-1} [y_i, x_j, z_j] c_{ij} + \sum_{i,j=1}^{r-1} [z_i, x_j, y_j] c_{ij} - \sum_{i,j=1}^{r-1} [z_i, x_j, x_j] c_{ij} \\ &- \sum_{i,j=1}^{r-1} [z_i, y_j, x_j] c_{ij} - \sum_{i,j=1}^{r-1} [x_i, x_j, z_j] c_{ij} - \sum_{i,j=1}^{r-1} [x_i, y_j, z_j] c_{ij}, \end{aligned}$$
for all spaces W and all $x, y, z \in [X, W]_\varphi$.

Proof. 1. We first consider inversivity. Using the method and notation of the proof of Proposition 6.4 with $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t) = L(x)$ and $\rho = (\rho_1, \rho_2, \dots, \rho_t) = R(x)$, we see that $\lambda_k = -x_k = \rho_k$, for all $k \neq r$,

$$\lambda_r = -x_r - \sum_{i,j=1}^{r-1} [x_i, \lambda_j, x_j] c_{ij} = -x_r - \sum_{i,j=1}^{r-1} \big(-[x_i, x_j, x_j]\big) c_{ij} \quad \text{and}$$

$$\rho_r = -x_r - \sum_{i,j=1}^{r-1} [\rho_i, x_i, \rho_j] c_{ij} = -x_r - \sum_{i,j=1}^{r-1} [x_i, x_j, x_j] c_{ij}.$$

Therefore, $\lambda_r = \rho_r$ if and only if

$$\big(-[x_i, x_j, x_j]\big) c_{ij} = [x_i, x_j, x_j] c_{ij},$$

for all $i, j < r$. This proves 1.

2. Next we consider power-associativity for φ. We have

$$\big(x +_\varphi (x +_\varphi x)\big)_k = 3x_k = \big((x +_\varphi x) +_\varphi x\big)_k \quad \text{if } k \neq r,$$
$$\big(x +_\varphi (x +_\varphi x)\big)_r = 3x_r + \sum_{i,j=1}^{r-1} [x_i, x_j, x_j]c_{ij} + \sum_{i,j=1}^{r-1} [2x_i, x_j, 2x_j]c_{ij}$$

and

$$\big((x +_\varphi x) +_\varphi x\big)_r = 3x_r + \sum_{i,j=1}^{r-1} [x_i, x_j, x_j]c_{ij} + \sum_{i,j=1}^{r-1} [x_i, 2x_j, x_j]c_{ij}.$$

Therefore $x +_\varphi (x +_\varphi x) = (x +_\varphi x) +_\varphi x$ if and only if $(4[x_i, x_j, x_j])c_{ij} = (2[x_i, x_j, x_j])c_{ij}$.

3. Next we consider the Moufang property. Let $x = (x_1, x_2, \ldots, x_r)$, $y = (y_1, y_2, \ldots, y_r)$ and $z = (z_1, z_2, \ldots, z_r)$ be elements in the loop $[X, W]_\varphi$. Then for $k \neq r$,

$$\big(\big(x +_\varphi (y +_\varphi z)\big) +_\varphi x\big)_k = 2x_k + y_k + z_k = \big((x +_\varphi y) +_\varphi (z +_\varphi x)\big)_k.$$

We also have

$$\begin{aligned}
\big((x +_\varphi (y +_\varphi z)) +_\varphi x\big)_r &= 2x_r + y_r + z_r + \sum_{i,j=1}^{r-1} [z_i, y_j, z_j]c_{ij} \\
&\quad + \sum_{i,j=1}^{r-1} [y_i + z_i, x_j, y_j + z_j]c_{ij} \\
&\quad + \sum_{i,j=1}^{r-1} [x_i, x_j + y_j + z_j, x_j]c_{ij} \\
&= 2x_r + y_r + z_r + U,
\end{aligned}$$

where by hypothesis and Corollary 3.2,

$$\begin{aligned}
U = {} & \sum_{i,j=1}^{r-1} [z_i, y_j, z_j]c_{ij} + \sum_{i,j=1}^{r-1} [y_i, x_j, y_j]c_{ij} + \sum_{i,j=1}^{r-1} [y_i, x_j, z_j]c_{ij} \\
& + \sum_{i,j=1}^{r-1} [z_i, x_j, y_j]c_{ij} + \sum_{i,j=1}^{r-1} [z_i, x_j, z_j]c_{ij} + \sum_{i,j=1}^{r-1} [x_i, x_j, x_j]c_{ij} \\
& + \sum_{i,j=1}^{r-1} [x_i, y_j, x_j]c_{ij} + \sum_{i,j=1}^{r-1} [x_i, z_j, x_j]c_{ij}.
\end{aligned}$$

Similarly,

$$\big((x +_\varphi y) +_\varphi (z +_\varphi x)\big)_r = 2x_r + y_r + z_r + V,$$

where

$$\begin{aligned}
V = {} & \sum_{i,j=1}^{r-1} [y_i, x_j, y_j]c_{ij} + \sum_{i,j=1}^{r-1} [x_i, z_j, x_j]c_{ij} + \sum_{i,j=1}^{k-1} [z_i, x_j, z_j]c_{ij} \\
& + \sum_{i,j=1}^{r-1} [z_i, x_j, x_j]c_{ij} + \sum_{i,j=1}^{r-1} [z_i, y_j, z_j]c_{ij} + \sum_{i,j=1}^{r-1} [z_i, y_j, x_j]c_{ij} \\
& + \sum_{i,j=1}^{r-1} [x_i, x_j, z_j]c_{ij} + \sum_{i,j=1}^{r-1} [x_i, x_j, x_j]c_{ij} + \sum_{i,j=1}^{r-1} [x_i, y_j, z_j]c_{ij} \\
& + \sum_{i,j=1}^{r-1} [x_i, y_j, x_j]c_{ij}.
\end{aligned}$$

Therefore, the loop $[X, W]_\varphi$ has the Moufang property if and only if

$$\sum_{i,j=1}^{r-1}[y_i, x_j, z_j]c_{ij} + \sum_{i,j=1}^{r-1}[z_i, x_j, y_j]c_{ij} - \sum_{i,j=1}^{r-1}[z_i, x_j, x_j]c_{ij} - \sum_{i,j=1}^{r-1}[z_i, y_j, x_j]c_{ij} - \sum_{i,j=1}^{r-1}[x_i, x_j, z_j]c_{ij} - \sum_{i,j=1}^{r-1}[x_i, y_j, z_j]c_{ij} = 0.$$

This completes the proof. □

Corollary 6.10. *Let $H_1^1(c_{ij}) = 0$ and $n_r \leqslant 3n_i + 6n_j - 9$ for all $i, j < r$.*

1. *Then φ is inversive if and only if $2([x_i, x_j, x_j]c_{ij}) = 0$, for all spaces W, all $x \in [X, W]_\varphi$ and all $i, j < r$.*
2. *Then φ is power-associative if and only if $2([x_i, x_j, x_j]c_{ij}) = 0$, for all spaces W and all $x \in [X, W]_\varphi$.*

Proof. The proofs of 1 and 2 are similar, so we only prove 1.

$$\begin{aligned} 0 &= \big([x_i, x_j, x_j] + \big(-[x_i, x_j, x_j]\big)\big)c_{ij} \\ &= [x_i, x_j, x_j]c_{ij} + \big(-[x_i, x_j, x_j]\big)c_{ij}, \end{aligned}$$

and so $(-[x_i, x_j, x_j])c_{ij} = -([x_i, x_j, x_j]c_{ij})$. Therefore by Proposition 6.9, φ is inversive if and only if $2([x_i, x_j, x_j]c_{ij}) = 0$, for all spaces W and all $x \in [X, W]_\varphi$. □

Remark 6.11. Similar to Remarks 4.3 and 6.5, we note that it is possible to give a larger upper bound for n_r in Proposition 6.9 and Corollary 6.10 by requiring the vanishing of more Hopf–Hilton invariants.

References

[1] M. Arkowitz, Co-H-spaces, in: Handbook of Algebraic Topology, North-Holland, New York, 1995, pp. 1143–1173.
[2] M. Arkowitz, M. Gutierrez, Comultiplications on free groups and wedge of circles, Trans. Amer. Math. Soc. 350 (4) (1998) 1663–1680.
[3] M. Arkowitz, D. Lee, Comultiplications on a wedge of two spheres, submitted for publication.
[4] M. Arkowitz, G. Lupton, Rational co-H-spaces, Comment. Math. Helv. 66 (1991) 79–108.
[5] M. Arkowitz, G. Lupton, Equivalence classes of homotopy-associative comultiplications of finite complexes, J. Pure Appl. Algebra 102 (1995) 109–136.
[6] M. Arkowitz, G. Lupton, Loop-theoretic properties of H-spaces, Math. Proc. Cambridge Philos. Soc. 110 (1991) 121–136.
[7] M. Golasinski, D.L. Goncalves, Comultiplications of the wedge of two Moore spaces, Colloq. Math. 76 (1998) 229–242.
[8] M. Golasinski, D.L. Goncalves, P. Wong, Generalization of Fox homotopy groups, Whitehead products and Gottlieb groups, Ukrainian Math. J. 57 (2005) 382–393.
[9] M. Golasinski, D.L. Goncalves, P. Wong, On Fox spaces and Jacobi identities, Math. J. Okayama Univ. 50 (2008) 161–176.
[10] P.J. Hilton, On the homotopy groups of the union of spheres, J. Lond. Math. Soc. 30 (2) (1955) 154–172.
[11] P. Hilton, G. Mislin, J. Roitberg, On co-H-spaces, Comment. Math. Helv. 53 (1978) 1–14.
[12] I.M. James, On H-spaces and their homotopy groups, Q. J. Math. 11 (1960) 161–179.
[13] C.W. Norman, Homotopy loops, Topology 2 (1963) 23–43.
[14] H. Toda, Composition Methods in Homotopy Groups of Spheres, Princeton University Press, Princeton, 1962.